

# On Serre functor in the category of strict polynomial functors

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## Abstract

We introduce and study a Serre functor in the category  $\mathcal{P}_d$  of strict polynomial functors over a field of positive characteristic. By using it we obtain the Poincaré duality formula for Ext-groups from [C3] in elementary way. We also show that the derived category of the category of affine strict polynomial functors in some cases carries the structure of Calabi-Yau category.

*Mathematics Subject Classification* (2010) 18A25, 18A40, 18G15. 20G15.

*Key words and phrases*: strict polynomial functor, Serre functor, Calabi-Yau category.

## 1 Introduction

In the present article we study a Serre functor in the category  $\mathcal{P}_d$  of strict polynomial functors of degree  $d$  over a field of positive characteristic. Although the existence of a Schur functor in our context follows from general theory, its interplay with various structures living on  $\mathcal{P}_d$  (Frobenius twist, affine subcategories, blocks) has some interesting consequences.

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\*The author was supported by the grant (NCN) 2011/01/B/ST1/06184.

Our paper can be naturally divided into two parts: Section 2 and Sections 3–5.

The objective of the first part is to give an elementary and self-contained account of the Serre functor  $\mathbf{S}$  in  $\mathcal{P}_d$  and to quickly obtain with the aid of  $\mathbf{S}$  the Poincaré duality formula from [C3]:

**Corollary 2.4** Let  $\lambda$  be a Young diagram of weight  $d$  which is single in its block (we call such a diagram and its block *basic*), let  $\mu$  be any Young diagram of weight  $p^i d$ . Let  $F_\lambda, F_\mu$  be the corresponding simple objects. Then

$$\mathrm{Ext}_{\mathcal{P}_{dp^i}}^s(F_\lambda^{(i)}, F_\mu) \simeq \mathrm{Ext}_{\mathcal{P}_{p^i d}}^{2d(p^i-1)-s}(F_\lambda^{(i)}, F_\mu)^*.$$

We believe that the approach to the Poincaré duality presented here is more intuitive than that taken in [C3] and that it is also better adapted for possible generalizations.

Then in Sections 3–5 we turn attention to the category  $\mathcal{P}_d^{afi}$  of  $i$ -affine strict polynomial functors of degree  $d$  and we introduce a (somewhat weaker version of) Serre functor on its derived category  $\mathcal{DP}_d^{afi}$ . The main goal of this part of the paper is to put the formula from Corollary 2.4 into a wider categorical context. In general, a Serre functor produces Poincaré duality in Ext-groups when it acts on some object as the shift functor. Indeed, we see in our Proposition 2.3 that this exactly happens for some Frobenius twisted strict polynomial functors (in fact, Corollary 2.4 is a formal consequence of Proposition 2.3). We provide a categorical interpretation of this phenomenon by finding certain subcategories of  $\mathcal{DP}_{dp^i}$  on which the Serre functor is isomorphic to the shift functor (such categories are called Calabi–Yau). In fact, we define these Calabi–Yau categories as the images of the basic blocks from  $\mathcal{P}_d$  in  $\mathcal{DP}_d^{afi}$  (we call these subcategories “basic (affine) semiblocks”). Appearing of  $\mathcal{DP}_d^{afi}$  here is quite natural, since it is a full triangulated subcategory of  $\mathcal{DP}_{dp^i}$  generated by the Frobenius twisted objects [C4, Theorem 5.1]. Thus we have succeeded in providing a categorical interpretation of the both assumptions in Corollary 2.4: we specialize to  $\mathcal{DP}_d^{afi}$  because  $F_\lambda^{(i)}$  is twisted and we restrict to the image of the block containing  $F_\lambda$  to take advantage of the fact that  $\lambda$  is basic.

Below we describe the contents of Sections 3–5 in some detail since the considerations there are much more involved than those in Section 2.

In Section 3 we study a Serre functor in  $\mathcal{DP}_d^{afi}$ . We start with reviewing basic properties of the categories  $\mathcal{P}_d^{afi}$  and  $\mathcal{DP}_d^{afi}$ . This is mainly recollecting some

facts from [C4] where these concepts were introduced and adapting them to a slightly more general setting of “multiple twists” in which we work in the present article. Then we proceed to define the Serre functor  $\mathbf{S}^{af_i}$  in  $\mathcal{DP}_d^{af_i}$ . However, we need to adapt this notion to the fact that  $\mathcal{DP}_d^{af_i}$  has infinite dimensional Hom-spaces. Hence, technically, we define “a weak Serre functor” (Definition 3.3) on  $\mathcal{DP}_d^{af_i}$ .

In Section 4 we introduce “the semiblock decomposition” of  $\mathcal{DP}_d^{af_i}$ . This is a collection of reflective subcategories of  $\mathcal{DP}_d^{af_i}$  indexed by the set of blocks in  $\mathcal{P}_d$ . They generate  $\mathcal{DP}_d^{af_i}$  but in contrast to genuine blocks they are not orthogonal. We believe that this structure deserves further investigation, in particular we conjecture that the semiblocks form a set of strata of certain stratification of  $\mathcal{DP}_d^{af_i}$ . In the present article we restrict ourself to introducing the affine derived Kan extension and Serre functor on the semiblocks. This is a non-trivial task due to the non-orthogonality of semiblocks.

In Section 5 we focus on the basic semiblocks, i.e. the subcategories of  $\mathcal{DP}_d^{af_i}$  which correspond to the blocks containing a single simple object. We show (Theorem 5.1) that they are Calabi–Yau categories thus providing the promised categorical interpretation of our Poincaré duality formula. We finish our paper by giving various explicit descriptions of basic semiblocks as categories of DG-modules over certain graded algebras (Proposition 5.5, Corollary 5.6) which should make them easier to handle.

Hence the main result of this part of the paper is Theorem 5.1 which shows that the basic semiblocks are Calabi–Yau categories. This is also important from a more general point of view. Namely, it seems that the basic semiblocks constitute sort of building blocks for  $\mathcal{DP}_{dp^i}$ , since many homological problems concerning strict polynomial functors can be reduced to statements about basic semiblocks. For example, the classical line of research started in [FS] and [FFSS] may be thought of as expanding of our understanding of the basic semiblock  $\mathcal{DP}_1^{af_i}$  onto the whole  $\mathcal{DP}_{p^i}$ . To be more specific, let us consider the fundamental problem of computing the Ext-groups between simple objects in  $\mathcal{P}_{dp^i}$ . Then one can hope that by using tools like the (Schur)–de Rham complex this problem can be reduced to that of computing

$$\mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(F_\lambda^{(i)}, F_\mu)$$

where  $\lambda$  is a basic Young diagram. This computation can be transferred by the affine derived Kan extension to the basic semiblock containing  $\lambda$ . Therefore a better understanding of the internal structure of the basic semiblocks seems to be an important step towards understanding  $\mathcal{DP}_{dp^i}$  in general.

## 2 Serre functor in $\mathcal{DP}_d$

Let  $\mathcal{P}_d$  be the category of strict polynomial functors of degree  $d$  over a fixed field  $\mathbf{k}$  of characteristic  $p > 0$  as defined in [FS]. For a finite dimensional  $\mathbf{k}$ -vector space  $U$  we define the strict polynomial functor  $S_{U*}^d \in \mathcal{P}_d$  by the formula

$$V \mapsto S^d(\mathrm{Hom}(U, V)).$$

We recall from [FS, Th. 2.10] the natural in  $U$  isomorphism

$$\mathrm{Hom}_{\mathcal{P}_d}(F, S_{U*}^d) \simeq F(U)^*$$

for any  $F \in \mathcal{P}_d$ . In fact, when we interpret  $\mathcal{P}_d$  as a functor category as is done e.g. in [FP, Sect. 3], this formula is just the Yoneda lemma. Hence later on we will refer to this formula as to the Yoneda lemma. It immediately follows from this formula that  $S_{U*}^d$  is injective and it was shown in [FS, Th. 2.10] that if  $\dim(U) \geq d$  then  $S_{U*}^d$  is a cogenerator of  $\mathcal{P}_d$ . Dually, we have a family of projective objects  $\Gamma_{U*}^d$  for which the Yoneda lemma gives the isomorphism

$$\mathrm{Hom}_{\mathcal{P}_d}(\Gamma_{U*}^d, F) \simeq F(U)$$

for any  $F \in \mathcal{P}_d$ .

Let  $\mathcal{DP}_d$  denote the bounded derived category of  $\mathcal{P}_d$ . We would like to define a Serre functor in the sense of [BK] on  $\mathcal{DP}_d$ . The problem of existence of Serre functor on a triangulated category is well understood [BV, RV]. The existence of Serre functor on  $\mathcal{DP}_d$  follows from the fact that it is equivalent to the bounded derived category of category of finitely generated modules over a finite dimensional algebra of finite homological dimension. When we translate the construction from [BV] into the context of functor categories we get

**Definition 2.1** *We define a functor  $\mathbf{S} : \mathcal{DP}_d \longrightarrow \mathcal{DP}_d$  by the formula*

$$\mathbf{S}(F)(V) := \mathrm{Hom}_{\mathcal{DP}_d}(S_{V*}^d, F).$$

This functor was also mentioned in [Kr]. In our article we start a systematic study of its properties. Our exposition is quite elementary, independent of generalities of [BV] and self-contained (with exception of a few places where we refer to [C1] to avoid repeating the same arguments). We start by warning the reader that we chose to work with less common left Serre functors which

are easier to describe in the framework of functor categories (although also the right Serre functor can be explicitly defined by using either the Kuhn duals, as we do in the proof of Theorem 2.2.4, or the monoidal structure on  $\mathcal{P}_d$  introduced in [Kr]). We gather below basic properties of  $\mathbf{S}$ . Parts 1, 4, 5 essentially follow from generalities, parts 2 and 3 are more specific to  $\mathcal{P}_d$ .

**Theorem 2.2** *The functor  $\mathbf{S}$  satisfies the following properties*

1. *There is a natural in  $U$  isomorphism in  $\mathcal{P}_d$*

$$\mathbf{S}(S_{U*}^d) \simeq \Gamma_{U*}^d.$$

2. *There is an isomorphism of functors*

$$\mathbf{S} \simeq \Theta \circ \Theta$$

where  $\Theta$  is the “Koszul duality” functor from [C1].

3. *For any  $F \in \mathcal{DP}_d$ ,  $G \in \mathcal{DP}_{d'}$  there are isomorphisms in respectively  $\mathcal{DP}_{dp^i}$ ,  $\mathcal{DP}_{d+d'}$*

- $\mathbf{S}(F^{(i)}) \simeq \mathbf{S}(F)^{(i)}[-2d(p^i - 1)]$
- $\mathbf{S}(F \otimes G) \simeq \mathbf{S}(F) \otimes \mathbf{S}(G).$

4.  *$\mathbf{S}$  is a self-equivalence of  $\mathcal{DP}_d$ .*

5. *There is a natural in  $F, G \in \mathcal{DP}_d$  isomorphism*

$$\mathrm{Hom}_{\mathcal{DP}_d}(F, G) \simeq \mathrm{Hom}_{\mathcal{DP}_d}(\mathbf{S}(G), F)^*,$$

that is,  $\mathbf{S}$  is a left Serre functor in the sense of [BK].

**Proof:** To see the first part we recall that since  $S_{U*}^d$  is injective, we have

$$\mathbf{S}(S_{U*}^d)(V) = \mathrm{Hom}_{\mathcal{DP}_d}(S_{V*}^d, S_{U*}^d) \simeq \mathrm{Hom}_{\mathcal{P}_d}(S_{V*}^d, S_{U*}^d) \simeq S_{V*}^d(U^*)^* \simeq \Gamma_{U*}^d(V)$$

by the Yoneda lemma.

In fact, [C1, Fact 2.2] can be easily extended to the “parameterized version”:

$$\Theta((S_\lambda)_{U*}) \simeq (W_\lambda)_{U*}.$$

From this we obtain the isomorphisms  $\Theta(S_{U*}^d) \simeq \Lambda_{U*}^d$  and  $\Theta(\Lambda_{U*}^d) \simeq \Gamma_{U*}^d$  which give the second part.

The formulae from part 3 follow from the analogous facts holding for  $\Theta$  [C1, Fact 2.6].

It is immediate that the “right Serre functor”  $\mathbf{S}_l := (-)^\# \circ \mathbf{S} \circ (-)^\#$  where  $(-)^\#$  is the Kuhn duality is the inverse of  $\mathbf{S}$  (c.f. [C1, Def. 2.3, Cor. 2.4]), which gives the fourth part.

In order to obtain the last part, it suffices to establish a natural in  $U$  isomorphism

$$\mathrm{Hom}_{\mathcal{DP}_d}(F, S_{U*}^d) \simeq \mathrm{Hom}_{\mathcal{DP}_d}(\mathbf{S}(S_{U*}^d), F)^*.$$

By the first part and injectivity of  $S_{U*}^d$  and projectivity of  $\Gamma_{U*}^d$  it reduces to

$$\mathrm{Hom}_{\mathcal{P}_d}(F, S_{U*}^d) \simeq \mathrm{Hom}_{\mathcal{P}_d}(\Gamma_{U*}^d, F)^*,$$

which follows from the Yoneda lemma.  $\blacksquare$

The fact that  $\mathbf{S}$  is a Serre functor can be used to obtain the Poincarè like formulae for the Ext-groups, provided that we are able to compute  $\mathbf{S}(F)$  in some interesting cases. We shall illustrate this idea by re-obtainng the most important example of the Poincarè duality formula for Ext-groups in  $\mathcal{P}_d$  established in [C3].

Let  $\lambda$  be the Young diagram of weight  $d$  which is a  $p$ -core. We recall that the blocks in  $\mathcal{P}_d$  are indexed by the  $p$ -core Young diagrams of weight  $d - jp$  and that the block labeled by  $\lambda$  contains only one simple object  $F_\lambda$ . We call such a Young diagram  $\lambda$  and the corresponding block *basic*.

**Proposition 2.3** *Let  $\lambda$  be a basic Young diagram. Then*

$$\mathbf{S}(F_\lambda^{(i)}) \simeq F_\lambda^{(i)}[-2d(p^i - 1)].$$

**Proof:** Since  $F_\lambda$  is single in its block, we have isomorphisms  $F_\lambda \simeq S_\lambda \simeq W_\lambda$ . Therefore

$$\Theta(F_\lambda) \simeq \Theta(S_\lambda) = W_\lambda^\sim.$$

Now, since also  $F_\lambda^\sim$  is single in its block, we obtain

$$\Theta(W_\lambda^\sim) \simeq \Theta(S_\lambda^\sim) = W_\lambda \simeq F_\lambda.$$

Thus we see that  $\mathbf{S}(F_\lambda) \simeq F_\lambda$  and our formula follows from Theorem 2.1.3.  $\blacksquare$

The Poincarè duality formula [C3, Example 3.3] is a formal consequence of Proposition 2.3.

**Corollary 2.4** *Let  $\lambda$  be a basic Young diagram,  $\mu$  be any Young diagram of weight  $p^i d$ , and  $F_\lambda, F_\mu$  be the corresponding simple objects. Then*

$$\mathrm{Ext}_{\mathcal{P}_{dp^i}}^s(F_\lambda^{(i)}, F_\mu) \simeq \mathrm{Ext}_{\mathcal{P}_{p^i d}}^{2d(p^i-1)-s}(F_\lambda^{(i)}, F_\mu)^*.$$

**Proof:** By applying the Kuhn duality (and using the fact that simple objects are self-dual) and then the Serre functor we obtain:

$$\begin{aligned} \mathrm{Ext}_{\mathcal{P}_{p^i d}}^s(F_\lambda^{(i)}, F_\mu) &\simeq \mathrm{Ext}_{\mathcal{P}_{p^i d}}^s(F_\mu, F_\lambda^{(i)}) = \mathrm{Hom}_{\mathcal{DP}_{dp^i}}(F_\mu, F_\lambda^{(i)}[s]) \simeq \\ &\mathrm{Hom}_{\mathcal{DP}_{dp^i}}(\mathbf{S}(F_\lambda^{(i)}[s]), F_\mu)^* \simeq \mathrm{Hom}_{\mathcal{DP}_{dp^i}}(F_\lambda^{(i)}[s - 2d(p^i - 1)], F_\mu)^* \simeq \\ &\mathrm{Ext}_{\mathcal{P}_{dp^i}}^{2d(p^i-1)-s}(F_\lambda^{(i)}, F_\mu)^*. \end{aligned}$$

■

In the next part of the paper we will describe a categorical phenomenon which is responsible for turning the Serre duality into the Poincarè duality when one deals with the Frobenius twists of strict polynomial functors.

### 3 Serre functor for affine functors

#### 3.1 Review of $i$ -affine functors

The category of affine strict polynomial functors  $\mathcal{P}_d^{af}$  was studied in [C4]. In the present paper we introduce its slight generalization: the category of  $i$ -affine strict polynomial functors  $\mathcal{P}_d^{af_i}$ , hence we start with reviewing its basic properties. Since all the proofs from [C4] still work in the present context, the reader is referred for them to [C4]. The only exception where we provide a full proof is Proposition 3.2 which was merely mentioned in [C4].

Let  $A_i := \mathbf{k}[x_1, x_2, \dots, x_i]/(x_1^p, x_2^p, \dots, x_i^p)$  for  $|x_j| = 2p^j$  and let  $\Gamma^d \mathcal{V}_{A_i}$  stands for the following graded  $\mathbf{k}$ -linear category. The objects of  $\Gamma^d \mathcal{V}_{A_i}$  are finite dimensional vector spaces, though we follow the convention taken in [C4, Section 2] and label them as  $V \otimes A_i$  where  $V$  is a finite dimensional vector space. The morphisms are given as

$$\mathrm{Hom}_{\Gamma^d \mathcal{V}_{A_i}}(V \otimes A_i, W \otimes A_i) := \Gamma^d(\mathrm{Hom}(V, W) \otimes A_i).$$

An  $i$ -affine strict polynomial functor of degree  $d$  is a graded functor from  $\Gamma^d \mathcal{V}_{A_i}$  to the category of  $\mathbf{Z}$ -graded bounded below finite dimensional in each

degree vector spaces (c.f. [C4, Section 2]). The  $i$ -affine strict polynomial functors of degree  $d$  form the  $\mathbf{k}$ -linear graded abelian category  $\mathcal{P}_d^{afi}$  with morphisms being the natural transformations. For any finite dimensional vector space  $U$  we have the representable  $i$ -affine strict polynomial functors of degree  $d$   $h^{U \otimes A_i}$  given by the formula

$$V \otimes A_i \mapsto \text{Hom}_{\Gamma^d \mathcal{V}_{A_i}}(V \otimes A_i, W \otimes A_i) = \Gamma^d(\text{Hom}(V, W) \otimes A_i),$$

and by the Yoneda lemma [C4, Prop. 2.2] we have

$$\text{Hom}_{\mathcal{P}_d^{af}}(h^{U \otimes A_i}, F) \simeq F(U \otimes A_i).$$

Similarly, we have the co-representable functor  $c_{U \otimes A_i}^*$  given by

$$V \otimes A \mapsto \text{Hom}_{\Gamma^d \mathcal{V}_A}(V \otimes A, U \otimes A)^*$$

where  $(-)^*$  stands for the graded  $\mathbf{k}$ -linear dual. This time the Yoneda lemma gives

$$\text{Hom}_{\mathcal{P}_d^{af}}(F, c_{U \otimes A}^*) \simeq F(U \otimes A)^*.$$

Analogously to the non-affine case,  $\mathcal{P}_d^{afi}$  is equivalent to some module category. Namely, let us define the  $i$ -affine Schur algebra  $S_{d,n}^{afi} := \Gamma^d(\text{End}(\mathbf{k}^n)) \otimes A_i$ . Then  $F(\mathbf{k}^n \otimes A_i)$  is naturally a graded  $S_{d,n}^{afi}$ -module and we have [C4, Prop. 2.5]

**Proposition 3.1** *If  $n \geq d$  then*

$$ev_n : \mathcal{P}_d^{afi} \longrightarrow S_{d,n}^{afi}\text{-mod}^{f+},$$

*where  $S_{d,n}^{af}\text{-mod}^{f+}$  is the category of bounded below finite dimensional in each degree graded  $S_{d,n}^{af}$ -modules, is an equivalence of graded abelian categories.*

The forgetful functor  $z : \Gamma^d \mathcal{V}_{A_i} \longrightarrow \Gamma^d \mathcal{V}$  induces an exact functor  $z^* : \mathcal{P}_d \longrightarrow \mathcal{P}_d^{afi}$  which has right and left adjoints  $t^*, h^* : \mathcal{P}_d^{afi} \longrightarrow \mathcal{P}_d$  (consult [C4, Sect. 2] on grading issue).

Much deeper is relation between  $\mathcal{P}_d^{afi}$  and  $\mathcal{P}_{dp^i}$ , since it only emerges at the level of derived categories. In order to develop homological algebra in  $\mathcal{P}_d^{afi}$  we regard it as a DG category (with the trivial differential) (see [K1], [K2], [C4, Section 3]). Then we consider the category of complexes over  $\mathcal{P}_d^{afi}$ , i.e. the category of graded functors from  $\Gamma^d \mathcal{V}_{A_i}$  to the category of



bounded below complexes of finite dimensional in each degree vector spaces. The derived category  $\mathcal{DP}_d^{af_i}$  is obtained from the category of complexes by inverting the class of quasiisomorphisms. This procedure can be conducted within the formalism of Quillen model categories. Namely the category of complexes over  $\mathcal{P}_d^{af_i}$  can be equipped with either of two model structures: the projective one in which every object is fibrant and  $h^{U \otimes A_i}$  are cofibrant and the injective one in which every object is cofibrant and  $c_{U \otimes A_i}^*$  are fibrant. In both cases the homotopy category is equivalent to  $\mathcal{DP}_d^{af_i}$ . The main result of [C4] is a construction of a full embedding

$$\mathbf{C}^{af_i} : \mathcal{P}_d^{af_i} \longrightarrow \mathcal{P}_{dp^i}$$

and its right adjoint

$$\mathbf{K}^{af_i} : \mathcal{P}_d^{af_i} \longrightarrow \mathcal{P}_{dp^i}$$

called the affine derived Kan extension. This adjunction is compatible with the adjunction  $\{\mathbf{C}, \mathbf{K}^r\}$  in the sense that we have isomorphisms of functors [C4, Theorem 5.1]:

$$\mathbf{K}^r \simeq t^* \circ \mathbf{K}^{af}, \quad \mathbf{C} \simeq \mathbf{C}^{af} \circ z^*.$$

We finish our review by discussing the compatibility of  $\{\mathbf{C}^{af}, \mathbf{K}^{af}\}$  with the Kuhn duality. This is a non-trivial problem, which was mentioned in [C4, Sect. 6]. In particular we shall use in the proof the main result of [C3]. We need it in the present article in order to connect the results of Sections 2 and 5.

We recall that  $\mathcal{DP}_d$  denotes the bounded derived category. We denote by  $\mathcal{DP}_d^+$  the derived category coming from the bounded below complexes. Let  $\mathcal{DP}_d^{af_i, b}$  be the smallest full triangulated subcategory of  $\mathcal{DP}_d^{af_i}$ , containing  $h^{U \otimes A_i}$  and closed under taking direct factors. In other words:  $\mathcal{DP}_d^{af_i, b}$  is the full subcategory of  $\mathcal{DP}_d^{af_i}$  consisting of all compact objects.

**Proposition 3.2** *We have the following isomorphisms of functors:*

1.

$$(-)^\# \circ \mathbf{K}^{af} \circ (-)^\# \simeq \mathbf{K}^{af}$$

as functors between  $\mathcal{DP}_{dp^i}^+$  and  $\mathcal{DP}_d^{af}$ ,

2.

$$(-)^\# \circ \mathbf{C}^{af} \circ (-)^\# \simeq \mathbf{C}^{af}$$

as functors between  $\mathcal{DP}_d^{af_i, b}$  and  $\mathcal{DP}_{dp^i}$ .

**Proof:** In fact the first part can be deduced from the proof of [C3, Theorem 2.1]. Let  $k : \Gamma^d \mathcal{V}_{A_i} \rightarrow \Gamma^d \mathcal{V}$  be the functor induced by the projection  $A_i \rightarrow \mathbf{k}$  and let  $k^* : \mathcal{DP}_d \rightarrow \mathcal{DP}_d^{af_i, b}$  be the functor induced by the precomposition with  $k$ . Then

$$\mathbf{K}^{af_i}(S_{U^*}^{dp^i}) = k^*(S_{U^{(i)*}}^d)$$

and

$$k^*(S_{U^{(i)*}}^d) = k^*(\Gamma_{U^{(i)*}}^d)[-2d(p^i - 1)]$$

c.f. [C4, Prop. 2.4.2]. Thus the isomorphism constructed in the proof of [C3, Theorem 2.1] can be interpreted as

$$\mathbf{K}^{af_i}(F^\#) \simeq \mathbf{K}^{af_i}(F)^\#$$

for any  $F \in \mathcal{DP}_{dp^i}$ . In order to extend this isomorphism to  $\mathcal{DP}_{dp^i}^+$  we observe that, since  $\mathbf{K}^{af_i}$  commutes with infinite direct colimits (because  $P^\bullet$  is a compact object in  $\mathcal{DP}_{dp^i}^+$ ), both left and right hand sides take direct colimits into codirect limits.

In order to obtain the second part we recall that

$$(h^{U \otimes A_i})^\# = c_{U^* \otimes A_i}^*[-2d(p^i - 1)] = z^*(S_{U^*}^d).$$

Hence

$$\begin{aligned} \mathbf{C}^{af_i}((h^{U \otimes A_i})^\#) &= \mathbf{C}^{af_i}(z^*(S_{U^*}^d)) = \mathbf{C}(S_{U^*}^d) = S_{U^*}^{d(i)} = \\ &= (\Gamma_{U^*}^{d(i)})^\# = (\mathbf{C}^{af_i}(h^{U \otimes A_i}))^\#, \end{aligned}$$

which gives the required isomorphism for any  $F \in \mathcal{DP}_d^{af_i, b}$ . This time, since  $\mathbf{C}^{af}$  does not commute with infinite codirect limits, the isomorphism cannot be extended to the whole  $\mathcal{DP}_d^{af_i}$ . Indeed, as it was observed in [C4, Section 6], e.g. for  $F = k^*(I) \in \mathcal{DP}_1^{af_1}$ , computing the both sides of the postulated isomorphism gives different results. ■

### 3.2 Serre functor in $\mathcal{DP}_d^{afi}$

In this subsection we introduce (a suitably modified version of) Serre functor in  $\mathcal{DP}_d^{afi}$ . In fact a genuine Serre functor exists only on  $\mathcal{DP}_d^{afi,b}$  but this is not very useful for us since the objects of  $\mathcal{DP}_d^{afi}$  rarely have cofibrant replacements in  $\mathcal{DP}_d^{afi,b}$ . On the other hand one cannot hope for existing of a Serre functor in  $\mathcal{DP}_d^{afi}$ , since it is not even a Hom-finite category. What we really have is the following weaker version of Serre functor:

**Definition 3.3** *Let  $\mathcal{C}$  be a  $\mathbf{k}$ -linear category. A  $\mathbf{k}$ -linear functor  $\mathbf{S} : \mathcal{C} \longrightarrow \mathcal{C}$  is a weak (left) Serre functor if:*

1. *There is a natural in  $X, Y$  isomorphism*

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{S}(X), Y) \simeq \mathrm{Hom}_{\mathcal{C}}(Y, X)^*$$

*whenever  $X$  or  $Y$  is compact.*

2.  *$\mathbf{S}$  is an auto-equivalence.*

For example, if  $\mathcal{C}$  is the bounded derived category of category of finitely generated modules over a finite dimensional algebra of finite homological dimension then by [BV]  $\mathcal{C}$  posses a Serre functor. In that case, it is easy to see that it extends to a weak Serre functor. However, our situation is quite different, since  $\mathcal{P}_d^{afi}$  is of infinite homological dimension. The crucial property of  $\mathcal{DP}_d^{afi}$  which makes a construction analogous to that used in Section 2 working is the following technical fact:

**Proposition 3.4** *For any finite dimensional space  $U$ ,  $c_{U \otimes A_i}^*$  is a compact object of  $\mathcal{DP}_d^{afi}$ , i.e. the functor  $\mathrm{Hom}_{\mathcal{DP}_d^{afi}}(c_{U \otimes A_i}^*, -)$  commutes with infinite direct colimits.*

Let  $P^\bullet$  be a finite projective resolution of  $S_{U^*}^d$  in  $\mathcal{P}_d$ . Then

$$h^*(P^\bullet) \simeq h^*(S_{U^*}^d) = c_{U \otimes A_i}^*.$$

Since  $h^* \simeq t^*[2d(p^i - 1)]$  by [C4, Prop. 2.4.5] and  $t^*$  preserves cofibrant objects,  $h^*(P^\bullet)$  is a cofibrant replacement of  $c_{U \otimes A_i}^*$ . To conclude the proof we observe that since  $P^\bullet$  is finite,  $h^*(P^\bullet)$  is compact. ■

Now we can define the affine Serre functor in a manner analogous to that used in Section 2.

**Definition 3.5** We define a functor  $\mathbf{S} : \mathcal{DP}_d^{af_i} \longrightarrow \mathcal{DP}_d^{af_i}$  by the formula

$$\mathbf{S}^{af_i}(F)(V \otimes A_i) := \text{Hom}_{\mathcal{DP}_d^{af_i}}(c_{V \otimes A_i}^*, F).$$

We collect the basic properties of  $\mathbf{S}^{af_i}$  which will be needed for the applications described in Section 5.

**Theorem 3.6** The functor  $\mathbf{S}^{af_i}$  satisfies the following properties:

1. There is a natural in  $U \otimes A_i$  isomorphism in  $\mathcal{P}_d^{af_i}$

$$\mathbf{S}^{af_i}(c_{U \otimes A_i}^*) \simeq h^{U \otimes A_i}.$$

2.  $\mathbf{S}^{af_i}$  is an auto-equivalence of  $\mathcal{DP}_d^{af_i}$ .
3.  $\mathbf{S}^{af_i}$  restricted to  $\mathcal{DP}_d^{af_i, b}$  is a left Serre functor and it is a weak left Serre functor on the whole  $\mathcal{DP}_d^{af_i}$ .
4. There are isomorphisms of functors

$$\mathbf{S}^{af_i} \circ z^*[2d(p^i - 1)] \simeq z^* \circ \mathbf{S}, \quad \mathbf{S} \circ t^* \simeq t^* \circ \mathbf{S}^{af_i}[2d(p^i - 1)],$$

$$\mathbf{S}^{af_i} \circ \mathbf{K}^{af} \simeq \mathbf{K}^{af} \circ \mathbf{S}, \quad \mathbf{S} \circ \mathbf{C}^{af_i} \simeq \mathbf{C}^{af} \circ \mathbf{S}^{af_i}.$$

**Proof:** Since  $c_{U \otimes A_i}^*$  is fibrant in the injective Quillen structure we get

$$\mathbf{S}^{af}(c_{U \otimes A_i}^*)(V \otimes A_i) \simeq \text{Hom}_{\mathcal{P}_d^{af_i}}(c_{V \otimes A_i}^*, c_{U \otimes A_i}^*) \simeq (c_{V \otimes A_i}^*(U \otimes A_i))^* =$$

$$\Gamma^d(\text{Hom}(U, V) \otimes A_i) = h^{U \otimes A_i}(V \otimes A_i).$$

In order to get the second part we define “the right affine Serre functor”  $\mathbf{S}_r^{af_i}$  by the formula

$$\mathbf{S}_r^{af_i}(F)(V \otimes A_i) := \text{Hom}_{\mathcal{P}_d^{af_i}}(F, h^{V \otimes A_i})^*.$$

Then by the computation analogous to that giving the first part we show that  $\mathbf{S}_r^{af_i}(h^{U \otimes A_i}) = c_{U \otimes A_i}^*$ . Thus the transformation  $id \longrightarrow \mathbf{S}^{af_i} \circ \mathbf{S}_r^{af_i}$  coming from the Yoneda lemma is an isomorphism for  $h^{U \otimes A_i}$ . Hence  $\mathbf{S}^{af_i}$  and  $\mathbf{S}_l^{af_i}$  are mutually inverse on  $\mathcal{DP}_d^{af_i, b}$ . Since, by Proposition 3.4,  $\mathbf{S}^{af_i}$  commutes with infinite direct colimits, it is an equivalence on the whole  $\mathcal{DP}_d^{af_i}$ .

In order to get the third part, again by Proposition 3.4, it suffices to establish a natural in  $F \in \mathcal{DP}_d^{afi}$  and  $U \otimes A_i$  isomorphism

$$\mathrm{Hom}_{\mathcal{DP}_d^{afi}}(F, c_{U \otimes A_i}^*) \simeq \mathrm{Hom}_{\mathcal{DP}_d^{afi}}(\mathbf{S}^{afi}(c_{U \otimes A_i}^*), F)^*,$$

which by the first part and the fact that  $h^{U \otimes A_i}$  is cofibrant and  $c_{U \otimes A_i}^*$  is fibrant reduces to the isomorphism

$$\mathrm{Hom}_{\mathcal{P}_d^{afi}}(F, c_{U \otimes A_i}^*) \simeq \mathrm{Hom}_{\mathcal{P}_d^{afi}}(h^{U \otimes A_i}, F)^*,$$

which follows from the Yoneda lemma.

In order to obtain the first isomorphism in part 4, we recall that  $z^*(\Gamma_{U^*}^d) = h^{U \otimes A_i}$  and  $z^*(S_{U^*}^d) = c_{U \otimes A_i}^*[-2d(p^i - 1)]$ . Hence we get natural in  $U$  isomorphisms:

$$\mathbf{S}^{afi} \circ z^*(S_{U^*}^d) = \mathbf{S}^{afi}(c_{U \otimes A_i}^*[-2d(p^i - 1)]) = h^{U \otimes A_i}[-2d(p^i - 1)]$$

and

$$z^* \circ \mathbf{S}(S_{U^*}^d) = z^*(\Gamma_{U^*}^d) = h^{U \otimes A_i}.$$

The second isomorphism follows from the facts that  $t^*(c_{U \otimes A_i}^*) = S_{(U \otimes A_i)^*}^d$  and  $t^*(h^{U \otimes A_i}) = \Gamma_{(U \otimes A_i)^*}^d[-2d(p^i - 1)]$ .

The proof of the last two isomorphisms is analogous to that of Proposition 3.2. The last formula holds on the whole  $\mathcal{DP}_d^{afi}$  because  $\mathbf{C}^{afi}$  commutes with infinite direct limits. ■

**Remark:** When we compose the isomorphisms from part 4, we obtain the formulae:

$$\mathbf{S} \circ \mathbf{K}^r \simeq \mathbf{K}^r \circ \mathbf{S}[2d(p^i - 1)], \quad \mathbf{S} \circ \mathbf{C} \simeq \mathbf{C} \circ \mathbf{S}[-2d(p^i - 1)],$$

from which, in particular, Theorem 2.2.3.1 follows. Thus we see that this shift phenomenon which produces the Poincaré duality is related to the scalar extension from  $\mathcal{P}_d$  to  $\mathcal{P}_d^{afi}$ .

## 4 Affine semiblocks

In this section we introduce certain subcategories of  $\mathcal{P}_d^{afi}$  which correspond to the blocks in  $\mathcal{P}_d$ . We call them semiblocks since they generate  $\mathcal{P}_d^{afi}$  and we

conjecture that  $\mathcal{P}_d^{af_i}$  is stratified by these subcategories. This structure may be interesting for its own but in the present article we are mainly interested in the Serre functor restricted to the semiblocks, since, as it will be shown in the next section, in certain cases it enjoys very special properties. We recall that the category  $\mathcal{P}_d$  admits decomposition into the blocks:

$$\mathcal{P}_d \simeq \mathcal{P}_{\lambda^1} \times \dots \times \mathcal{P}_{\lambda^s}$$

and the set of blocks is indexed by the family  $\lambda^1, \dots, \lambda^s$  of  $p$ -core Young diagrams of weight  $d - pj$  for some  $j \geq 0$ . By the Yoneda lemma, there is the corresponding decomposition of the bifunctor  $(V, W) \mapsto \Gamma^d(\text{Hom}(W, V))$  into the “block bifunctors”:

$$\Gamma^d(\text{Hom}(W, V)) \simeq B_{\lambda^1}(V, W) \oplus \dots \oplus B_{\lambda^s}(V, W).$$

The Cauchy decomposition [ABW, Th. III.1.4] provides the filtration of bifunctor  $\Gamma^d(\text{Hom}(W, V))$  with the associated object

$$\bigoplus_{\mu \in Y_d} W_\mu(V) \otimes W_\mu(W^*)$$

where  $Y_d$  stands for the set of Young diagrams of weight  $d$ . Hence each  $B_\lambda(V, W)$  has the filtration with the associated object

$$\bigoplus_{\mu \in Y_{\lambda^j}} W_\mu(V) \otimes W_\mu(W^*)$$

where  $Y_\lambda$  is the set of Young diagrams of degree  $d$  belonging to the block labeled by  $\lambda^j$ .

Moreover, the bifunctor  $B_{\lambda^j}$  can be used to form the category  $B_{\lambda^j}\mathcal{V}$  whose objects are finite vector spaces and

$$\text{Hom}_{B_{\lambda^j}\mathcal{V}}(V, W) := B_{\lambda^j}(W, V).$$

Then the category  $\mathcal{P}_{\lambda^j}$  can be identified with the category of  $\mathbf{k}$ -linear functors from  $B_{\lambda^j}\mathcal{V}$  to the category of finite dimensional vector spaces over  $\mathbf{k}$ . The main objective of the present section is to define the affine counterpart of  $\mathcal{P}_\lambda$ , relate it to  $\mathcal{P}_\lambda$  and  $\mathcal{P}_{dp^i}$ , and equip it with a Serre functor.

Let us fix a  $p$ -core Young diagram  $\lambda$  of weight  $|\lambda| = d - pj$  and let  $B_\lambda^{(i)}$

denote the bifunctor  $(V, W) \mapsto B_\lambda(V^{(i)}, W)$ . We introduce the graded category  $B_\lambda \mathcal{V}_{A_i}$  with the objects being finite dimensional vector spaces and the morphisms given by the formula

$$\mathrm{Hom}_{B_\lambda \mathcal{V}}(V \otimes A_i, V' \otimes A_i) := \mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(B_\lambda^{(i)}(-, V'), B_\lambda^{(i)}(-, V))$$

where we choose to label the objects by  $V \otimes A_i$  in order to make our terminology coherent with that used in Section 3 and [C4]. Thanks to the Collapsing Conjecture [C3, Cor. 3.7] the Hom-spaces in  $B_\lambda \mathcal{V}_{A_i}$  admit a more explicit description. Namely, we have natural in  $V, V'$  isomorphisms

$$\begin{aligned} \mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(B_\lambda^{(i)}(-, V'), B_\lambda^{(i)}(-, V)) &\simeq \mathrm{Ext}_{\mathcal{P}_d}^*(B_\lambda(-, V'), B_\lambda(- \otimes A_i, V)) \simeq \\ &B_\lambda(V' \otimes A_i, V). \end{aligned}$$

With this description the composition in  $B_\lambda \mathcal{V}_{A_i}$  is given as the composite of the scalar extension:

$$B_\lambda(V'' \otimes A_i, V') \otimes B_\lambda(V' \otimes A_i, V) \longrightarrow B_\lambda(V'' \otimes A_i \otimes A_i, V' \otimes A_i) \otimes B_\lambda(V' \otimes A_i, V),$$

the composition in  $B_\lambda \mathcal{V}$ :

$$B_\lambda(V'' \otimes A_i \otimes A_i, V' \otimes A_i) \otimes B_\lambda(V' \otimes A_i, V) \longrightarrow B_\lambda(V'' \otimes A_i \otimes A_i, V)$$

and the morphism

$$B_\lambda(V'' \otimes A_i \otimes A_i, V) \longrightarrow B_\lambda(V'' \otimes A_i, V)$$

induced by the multiplication  $A_i \otimes A_i \longrightarrow A_i$ . We then define  $\mathcal{P}_\lambda^{af_i}$  as the category of graded  $\mathbf{k}$ -linear functors from  $B_\lambda \mathcal{V}_{A_i}$  to the category of  $\mathbf{Z}$ -graded bounded below finite dimensional in each degree vector spaces.

The category  $\mathcal{P}_\lambda^{af_i}$  shares with  $\mathcal{P}_d^{af_i}$  its basic properties. In particular we have the representable functor  $h_\lambda^{U \otimes A_i}$  in  $\mathcal{DP}_\lambda^{af_i}$  given explicitly by the formula

$$h_\lambda^{U \otimes A_i}(V) := \mathrm{Hom}_{B_\lambda \mathcal{V}_{A_i}}(U \otimes A_i, V \otimes A_i),$$

the corepresentable functor  $c_{\lambda, U \otimes A_i}^*$  and the block affine Kuhn duality. Also the analog of Proposition 3.1 holds. Let us call the block affine Schur algebra the graded algebra

$$\mathrm{Hom}_{B_\lambda \mathcal{V}_{A_i}}(\mathbf{k}^d \otimes A_i, \mathbf{k}^d \otimes A_i) \simeq B_\lambda(\mathbf{k}^d \otimes A_i, \mathbf{k}^d).$$

Then

**Proposition 4.1** *The evaluation functor  $F \mapsto F(\mathbf{k}^d \otimes A_i)$  gives an equivalence of graded categories*

$$\mathcal{P}_\lambda^{afi} \simeq B_\lambda(\mathbf{k}^d \otimes A_i, \mathbf{k}^d) - grmod.$$

At last, the adjunction  $\{z^*, t^*\}$  between  $\mathcal{P}_d$  and  $\mathcal{P}_d^{afi}$  clearly extends to the adjunction  $\{z_\lambda^*, t_\lambda^*\}$  between  $\mathcal{P}_\lambda$  and  $\mathcal{P}_\lambda^{afi}$ .

On the other hand, when we try to decompose  $\mathcal{P}_d^{afi}$ , into the product of  $\mathcal{P}_{\lambda'}^{afi}$  we face a problem that for  $\lambda \neq \lambda'$ ,  $\mathcal{P}_\lambda^{afi}$  and  $\mathcal{P}_{\lambda'}^{afi}$  are not orthogonal as subcategories of  $\mathcal{P}_d^{afi}$ . We will come back to this observation later, since it is best comprehensible at the level of derived categories.

Now we turn to describing relation between  $\mathcal{P}_\lambda^{afi}$  and  $\mathcal{P}_d^{afi}$  more precisely. Let

$$i_\lambda : B_\lambda(V \otimes A_i, W) \longrightarrow \Gamma^d(\text{Hom}(W, V \otimes A_i))$$

be the natural embedding and

$$\pi_\lambda : \Gamma^d(\text{Hom}(W, V \otimes A_i)) \longrightarrow B_\lambda(V \otimes A_i, W)$$

be the natural projection. Then the composite  $\epsilon_\lambda := i_\lambda \circ \pi_\lambda$  can be thought of as an idempotent endofunctor on  $\Gamma^d \mathcal{V}_{A_i}$  (being the identity on the objects). Thus the category  $B_\lambda \mathcal{V}_{A_i}$  can be identified with the category  $\epsilon_\lambda(\Gamma^d \mathcal{V}_{A_i})\epsilon_\lambda$  whose objects are those of  $\Gamma^d \mathcal{V}_{A_i}$  but

$$\text{Hom}_{\epsilon_\lambda(\Gamma^d \mathcal{V}_{A_i})\epsilon_\lambda}(V, V') := \epsilon_\lambda(\text{Hom}_{\Gamma^d \mathcal{V}_{A_i}}(V, V'))\epsilon_\lambda.$$

Then the assignment

$$(V, V') \mapsto \epsilon_\lambda(\text{Hom}_{\Gamma^d \mathcal{V}_{A_i}}(V, V'))$$

defines a  $\Gamma^d \mathcal{V}_{A_i} \times B_\lambda \mathcal{V}_{A_i}$ -bimodule in the terminology of [K1, Sect. 6]. Hence we get a pair of functors  $j_\lambda^*, j_{\lambda*}$  which satisfy the following properties.

**Proposition 4.2** *1. The functor  $j_{\lambda*}$  is right adjoint to  $j_\lambda^*$ .*

*2. The functor  $j_\lambda^* : \mathcal{P}_\lambda^{afi} \longrightarrow \mathcal{P}_d^{afi}$  is a full embedding.*

**Proof:** The adjunction follows from the machinery of standard functors developed in [K1, Sect. 6]. The full embedding follows from the fact that  $j_\lambda^* \circ j_{\lambda*} \simeq id$ . ■

Let us remark that Proposition 4.2 may be thought of as a categorification



of [CPS, Prop. 2.1]. This explains our choice of notations with  $j^*, j_*$  instead of  $H_X, T_X$  used in [K1]. In fact we could derive Proposition 4.2 directly from [CPS, Prop. 2.1] by invoking our Proposition 4.1 but we prefer to consistently work in functor categories. We also mention that Proposition 4.2 carries over to the level of derived categories which was the main objective of [K1] and [CPS] and which will be discussed in the next paragraph.

Namely, we define  $\mathcal{DP}_\lambda^{af}$  as the derived category of DG-category  $\mathcal{P}_\lambda^{af}$  in the manner analogous to that in Section 3. The adjunctions  $\{z_\lambda^*, t_\lambda^*\}$  and  $\{j_\lambda^*, j_{\lambda*}\}$  carry over to the derived categories and, as we have already mentioned, the analog of the second part of Proposition 4.2 holds, i.e. we have the full embedding

$$j_\lambda^* : \mathcal{DP}_\lambda^{af} \longrightarrow \mathcal{DP}_d^{af},$$

which allows us to regard  $\mathcal{DP}_\lambda^{af}$  as a full subcategory of  $\mathcal{DP}_d^{af}$ . Then it is clear that our construction is compatible with the scalar extension from  $\mathcal{DP}_d$  to  $\mathcal{DP}_d^{af}$ :

**Proposition 4.3** *There are isomorphisms of functors:*

$$t^* \circ j_\lambda^* \simeq b_\lambda^* \circ t^*, \quad z^* \circ b_{\lambda*} \simeq j_{\lambda*} \circ z^*,$$

where  $b_\lambda^*$  and  $b_{\lambda*}$  are induced respectively by the embedding of and the projection onto the block.

Now we would like to construct a block version of the affine derived Kan extension in order to relate  $\mathcal{DP}_\lambda^{af}$  to  $\mathcal{DP}_{p^i d}$ . For this we need an analog of the formality result [C4, Th. 4.2]. Let  $X_\lambda$  be a projective resolution of  $B_\lambda^{(i)}$  in  $\mathcal{P}_{dp^i}^d$ . We introduce a DG category  $\Gamma^d \mathcal{V}_{X_\lambda}$  with the objects being finite dimensional vector spaces and

$$\mathrm{Hom}_{\Gamma^d \mathcal{V}_{X_\lambda}}(V, V') := \mathrm{Hom}_{\mathcal{P}_{dp}}(X_\lambda(-, V'), X_\lambda(-, V)).$$

Then  $B_\lambda \mathcal{V}_{A_i}$  is clearly the cohomology category of  $\Gamma^d \mathcal{V}_{X_\lambda}$  but we have a much stronger result (c.f. [C4, Th. 4.2]):

**Proposition 4.4** *The identity on the objects extends to an equivalence of DG categories  $\phi_\lambda : B_\lambda \mathcal{V}_{A_i} \simeq \Gamma^d \mathcal{V}_{X_\lambda}$ .*

**Proof:** Since  $\Gamma^d(I^{(i)} \otimes I^*) \simeq B_\lambda^{(i)} \oplus B'$ , we can obtain  $X$ , the projective resolution of  $\Gamma^d(I^{(i)} \otimes I^*)$ , as the direct sum  $X = X_\lambda \oplus X'$  of projective resolutions of  $B_\lambda^{(i)}$  and  $B'$ . Let

$$i_\lambda : \mathrm{Ext}_{\mathcal{P}_{dp}}^*(B_\lambda^{(i)}(-, V'), B_\lambda^{(i)}(-, V)) \longrightarrow \mathrm{Ext}_{\mathcal{P}_{dp}}^*(\Gamma^d((-)^{(i)}, V'), \Gamma^d((-)^{(i)}, V))$$

be the embedding induced by the decomposition  $\Gamma^d(I^{(i)} \otimes I^*) \simeq B_\lambda^{(i)} \oplus B'$  (we have already encountered this embedding when constructing the idempotent functor  $\epsilon_\lambda$ ). Let us define similarly the projection

$$\tilde{\pi}_\lambda : \text{Hom}_{\mathcal{P}_{dp}}(X(-, V'), X(-, V)) \longrightarrow \text{Hom}_{\mathcal{P}_{dp}}(X_\lambda(-, V')X_\lambda(-, V)).$$

We define  $\phi_\lambda : B_\lambda \mathcal{V}_{A_i} \longrightarrow \Gamma^d \mathcal{V}_{X_\lambda}$  as the composite  $\phi_\lambda := \tilde{\pi}_\lambda \circ \phi \circ i_\lambda$  where  $\phi : \Gamma^d \mathcal{V}_{A_i} \longrightarrow \Gamma^d \mathcal{V}_X$  is the transformation from [C4, Theroem 4.2] or rather its multitwist analog (in fact this generalization is not entirely trivial, we refer the reader to [C5, Theorem 3.1] where an analogous construction is conducted in even greater generality). Then the fact that  $\phi_\lambda$  is an equivalence easily follows from the fact that  $\phi$  is an equivalence and that it commutes with the idempotent  $\epsilon_\lambda := i_\lambda \circ \pi_\lambda$  and its  $\Gamma^d \mathcal{V}_{X_\lambda}$ -analog  $\tilde{\epsilon}_\lambda := \tilde{i}_\lambda \circ \tilde{\pi}_\lambda$ . ■

Thanks to Proposition 4.4 we are able to construct the block affine derived Kan extension. We summarize its basic properties below

**Proposition 4.5** *There exist functors  $\mathbf{C}_\lambda^{afi} : \mathcal{DP}_\lambda^{afi} \longrightarrow \mathcal{DP}_{dp^i}$  and  $\mathbf{K}_\lambda^{afi} : \mathcal{DP}_{dp^i} \longrightarrow \mathcal{DP}_\lambda^{afi}$  satisfying the following properties:*

1.  $\mathbf{K}_\lambda^{afi}$  is right adjoint to  $\mathbf{C}_\lambda^{afi}$ .
2.  $\mathbf{C}_\lambda^{afi}$  is a full embedding.
3. The functors  $\mathbf{C}_\lambda^{afi}$  (restricted to  $\mathcal{DP}_\lambda^{afi,b}$ ) and  $\mathbf{K}_\lambda^{afi}$  commute with the Kuhn duality.
4. There are isomorphisms of functors:

$$\mathbf{C}^{afi} \circ j_{\lambda*} \simeq \mathbf{C}_\lambda^{afi}, \quad j_\lambda^* \circ \mathbf{K}^{afi} \simeq \mathbf{K}_\lambda^{afi}.$$

The proofs of parts 1, 2, 3 are analogous to those of [C4, Th. 5.1] and our Proposition 3.2. The compatibilty formula in part 3 follows immediately from the constuction of the considered functors. ■

Having at our disposal the block affine drived Kan extension we can offer a better explanation of the phenomenon of non-orthogonality of semiblocks. Namely let us take  $F \in \mathcal{P}_\lambda$ ,  $G \in \mathcal{P}_{\lambda'}$  for  $\lambda \neq \lambda'$ . Then

$$\text{Hom}_{\mathcal{DP}_d^{afi}}^*(j_\lambda^*(z_\lambda^*(F)), j_{\lambda'}^*(z_{\lambda'}^*(G))) \simeq \text{Hom}_{\mathcal{DP}_{dp^i}}^*(\mathbf{C}_\lambda^{afi}(j_\lambda^*(z_\lambda^*(F))), \mathbf{C}_{\lambda'}^{afi}(j_{\lambda'}^*(z_{\lambda'}^*(G)))) \simeq$$

$$\simeq \text{Ext}_{\mathcal{P}_{dp^i}}^*(F^{(i)}, G^{(i)})$$

and the latter Ext-groups may well be non-trivial. Thus we see that the reason for the non-orthogonality of semiblocks is simply that the Frobenius twist transfers all the blocks from  $\mathcal{P}_d$  into the single (principal) block in  $\mathcal{P}_{dp^i}$ . Still, the block decomposition on  $\mathcal{P}_d$  when pushed to  $\mathcal{P}_d^{af_i}$  generates certain structure on the latter category. It seems that we have on  $\mathcal{P}_d^{af_i}$  (and its derived category) a stratification with the set of strata indexed by the blocks of  $\mathcal{P}_d$  but we defer a deeper study of semiblock structure to a future work. We finish this section by endowing the category  $\mathcal{DP}_\lambda^{af_i}$  with a Serre functor. This may be achieved by a construction analogous to that given in the global (affine) case.

**Definition 4.6** *We define the block affine Serre functor  $\mathbf{S}_\lambda^{af_i} : \mathcal{DP}_\lambda^{af_i} \longrightarrow \mathcal{DP}_\lambda^{af_i}$  by the formula*

$$\mathbf{S}_\lambda^{af_i}(F)(V \otimes A_i) := R\text{Hom}_{\mathcal{P}_\lambda^{af_i}}(c_{\lambda,V}^*, F).$$

Then the block analog of Proposition 3.5 holds

**Theorem 4.7** *The functor  $\mathbf{S}_\lambda^{af_i}$  satisfies the following properties:*

1. *There is a natural in  $U \otimes A_i$  isomorphism in  $\mathcal{P}_\lambda^{af_i}$*

$$\mathbf{S}_\lambda^{af_i}(c_{\lambda,U \otimes A_i}^*) \simeq h_\lambda^{U \otimes A_i}.$$

2.  *$\mathbf{S}_\lambda^{af_i}$  is an auto-equivalence of  $\mathcal{DP}_d^{af_i}$ .*
3.  *$\mathbf{S}_\lambda^{af_i}$  restricted to  $\mathcal{DP}_\lambda^{af_i,b}$  is a left Serre functor and it is a weak left Serre functor on the whole  $\mathcal{DP}_\lambda^{af_i}$ .*
4. *There are isomorphisms of functors*

$$\mathbf{S}_\lambda^{af_i} \circ z_\lambda^*[2d(p^i - 1)] \simeq z_\lambda^* \circ \mathbf{S}_\lambda, \quad \mathbf{S}_\lambda \circ t_\lambda^* \simeq t_\lambda^* \circ \mathbf{S}_\lambda^{af_i}[2d(p^i - 1)],$$

$$\mathbf{S}_\lambda^{af_i} \circ \mathbf{K}_\lambda^{af} \simeq \mathbf{K}_\lambda^{af} \circ \mathbf{S}_\lambda, \quad \mathbf{S}_\lambda \circ \mathbf{C}_\lambda^{af_i} \simeq \mathbf{C}_\lambda^{af} \circ \mathbf{S}_\lambda^{af_i},$$

where  $\mathbf{S}_\lambda$  is  $\mathbf{S}$  restricted to the block  $\mathcal{DP}_\lambda$ .

The proof of Theorem 3.6 carries over to the current situation.

## 5 Calabi–Yau structure on basic affine blocks

In this section we show that the affine Serre functor when restricted to certain semiblocks in  $\mathcal{DP}_d^{afi}$  is isomorphic to the shift functor.

We recall that a block in  $\mathcal{P}_d$  is called basic if it contains a single simple object. Hence the basic blocks are indexed by  $p$ -core Young diagrams of weight  $d$  and we also call such Young diagrams basic. So, let us fix a basic Young diagram  $\lambda$ . Then  $S_\lambda \simeq W_\lambda \simeq F_\lambda$ . Moreover,  $S_\lambda$  is injective and projective and every object of  $\mathcal{P}_\lambda$  is a direct sum of  $S_\lambda$ , therefore the category  $\mathcal{P}_\lambda$  is semisimple.

We recall that a triangulated category  $\mathcal{T}$  with a Serre functor  $\mathbf{S}_\mathcal{T}$  is called *Calabi–Yau of dimension  $n$*  if there is an isomorphism of functors  $\mathbf{S}_\mathcal{T} \simeq \text{id}[n]$ . Then we call a triangulated category  $\mathcal{T}$  *weak Calabi–Yau of dimension  $n$*  if it has a weak Serre functor  $\mathbf{S}_\mathcal{T}$  such that  $\mathbf{S}_\mathcal{T} \simeq \text{id}[n]$ .

**Theorem 5.1** *For any basic Young diagram  $\lambda$ , the category  $\mathcal{DP}_\lambda^{afi,b}$  is Calabi–Yau of dimension  $2d(p^i - 1)$ , the category  $\mathcal{DP}_\lambda^{afi}$  is weak Calabi–Yau of dimension  $2d(p^i - 1)$ .*

**Proof:** The theorem is a formal consequence of the following properties of the bifunctor  $B_\lambda$  (the crucial second property is specific to basic blocks).

**Lemma 5.2** *There are the following isomorphisms of bifunctors:*

1.  $B_\lambda(V, W \otimes A_i) \simeq B_\lambda(V \otimes A_i^*, W)$  for any  $\lambda$ .
2.  $B_\lambda(V, W) \simeq S_\lambda(V) \otimes S_\lambda(W^*)$  for basic  $\lambda$ .

**Proof of the Lemma** We recall that

$$B_\lambda(V, W) = \text{Hom}_{\mathcal{P}_d}(B_\lambda(-, W), B_\lambda(-, V))$$

and a general fact that

$$\text{Hom}_{\mathcal{P}_d}(F(- \otimes X), G) \simeq \text{Hom}_{\mathcal{P}_d}(F, G(- \otimes X^*))$$

for any graded space  $X$  and  $F, G \in \mathcal{P}_d$ . This gives the first isomorphism. The second isomorphism immediately follows from the Cauchy decomposition and the fact that  $S_\lambda \simeq W_\lambda$  for basic  $\lambda$ . ■

We recall that we deal with left Serre functors, hence we should show that

$$\mathbf{S}_\lambda^{af_i} \simeq id[-2d(p^i - 1)].$$

Since

$$\mathbf{S}_\lambda^{af_i}(F)(V \otimes A_i) := \mathrm{RHom}_{\mathcal{P}_\lambda^{af_i}}(c_{\lambda,V}^*, F)$$

and by the Yoneda lemma

$$F(V \otimes A_i) := \mathrm{RHom}_{\mathcal{P}_\lambda^{af_i}}(h_\lambda^V, F),$$

it suffices to find a natural in  $V$  isomorphism

$$c_{\lambda,V}^* \simeq h_\lambda^V[2d(p^i - 1)].$$

On one hand we have:

$$h_\lambda^V(W) = \mathrm{Hom}_{B_\lambda \vee_{A_i}}(V, W) = (B_\lambda(W \otimes A_i, V)) \simeq S_\lambda(W \otimes A_i) \otimes S_\lambda(V^*),$$

on the other hand:

$$\begin{aligned} c_{\lambda,V}^*(W) &= (\mathrm{Hom}_{B_\lambda \vee_{A_i}}(W, V))^* = (B_\lambda(V \otimes A_i, W))^* \simeq (B_\lambda(V, W \otimes A_i^*))^* \simeq \\ &= (S_\lambda(V) \otimes S_\lambda(W^* \otimes A_i))^* = S_\lambda(V^*) \otimes S_\lambda(W \otimes A_i^*). \end{aligned}$$

Since  $A_i^* \simeq A_i[2(p^i - 1)]$  we have an isomorphism of functors

$$S_\lambda(- \otimes A_i^*) \simeq S_\lambda(- \otimes A_i)[2d(p^i - 1)]$$

which completes the proof.  $\blacksquare$

**Corollary 5.3** *The category  $\mathcal{DP}_1^{af_i,b}$  is Calabi–Yau of dimension  $2(p^i - 1)$  and  $\mathcal{DP}_1^{af_i}$  is weak Calabi–Yau of dimension  $2(p^i - 1)$ .*

**Proof:** The corollary follows from Theorem 5.1 and the fact that  $\mathcal{P}_1$  consists of a single block which is obviously basic.  $\blacksquare$

This fact has the following global generalization.

**Proposition 5.4** *For any  $d < p$ , the category  $\mathcal{DP}_d^{af_i,b}$  is Calabi–Yau of dimension  $2d(p^i - 1)$  and  $\mathcal{DP}_d^{af_i}$  is weak Calabi–Yau of dimension  $2d(p^i - 1)$ .*

**Proof:** In fact for  $d < p$  all the blocks in  $\mathcal{P}_d$  are basic but since  $\mathcal{P}_d^{af_i}$  is not a product of its affine semiblocks, our statement cannot be directly deduced from Theorem 5.1. Instead one can repeat the proof of Theorem 5.1 in the present context. The crucial fact is that  $\Gamma^d \simeq S^d$  if  $d < p$ . We leave the straightforward details to the reader. ■

As we have said in the Introduction, the Calabi–Yau structure on  $\mathcal{DP}_\lambda^{af_i}$  provides sort of categorical interpretation of the Poincaré duality. Hence it is not surprising that one can deduce Corollary 2.4 from Theorem 5.1 (and the compatibility of the (block ) affine derived Kan extension with the Kuhn duality).

Namely, by the block affine derived Kan extension we obtain

$$\mathrm{Ext}_{\mathcal{P}_{p^i d}}^s(F_\mu^{(i)}, F_\lambda) \simeq \mathrm{Hom}_{\mathcal{DP}_\lambda^{af_i}}(z_\lambda^*(F_\lambda), \mathbf{K}_\lambda^{af_i}(F_\mu)[s]).$$

Then we apply the Calabi–Yau isomorphism (we emphasize that we need “the weak Calabi–Yau structure” here, since  $\mathbf{K}^{af_i}$  does not preserve compact objects)

$$\mathrm{Hom}_{\mathcal{DP}_\lambda^{af_i}}(z_\lambda^*(F_\lambda), \mathbf{K}_\lambda^{af_i}(F_\mu)[s]) \simeq \mathrm{Hom}_{\mathcal{DP}_\lambda^{af_i}}(\mathbf{K}_\lambda^{af_i}(F_\mu)[s-2d(p^i-1)], z_\lambda^*(F_\lambda))^*.$$

Next we apply the Kuhn duality and use the fact that it commutes with  $z^*$  and  $\mathbf{K}_\lambda^{af_i}$

$$\mathrm{Hom}_{\mathcal{DP}_\lambda^{af_i}}(\mathbf{K}_\lambda^{af_i}(F_\mu)[s-2d(p^i-1)], z_\lambda^*(F_\lambda))^* \simeq \mathrm{Hom}_{\mathcal{DP}_\lambda^{af_i}}(z_\lambda^*(F_\lambda^\#)[s-2d(p^i-1)], \mathbf{K}_\lambda^{af_i}(F_\mu^\#))^*.$$

At last we come back to  $\mathcal{DP}_{dp^i}$ :

$$\mathrm{Hom}_{\mathcal{DP}_\lambda^{af_i}}(z_\lambda^*(F_\lambda^\#)[s-2d(p^i-1)], \mathbf{K}_\lambda^{af_i}(F_\mu^\#))^* \simeq \mathrm{Hom}_{\mathcal{DP}_{dp^i}}(F_\lambda^{(i)\#}[s-2d(p^i-1)], F_\mu^\#))^*$$

and by using selfduality of simples we finally obtain our formula

$$\mathrm{Hom}_{\mathcal{DP}_{dp^i}}(F_\lambda^{(i)\#}[s-2d(p^i-1)], F_\mu^\#))^* \simeq \mathrm{Ext}_{\mathcal{P}_{dp^i}}^{2d(p^i-1)-s}(F_\lambda^{(i)}, F_\mu)^*.$$

Of course this approach is technically much more involved than that taken in Section 2, but it shows how classical and affine phenomena are related and also explains why we insist on considering weak Serre functors.

We finish our paper by providing description of  $\mathcal{P}_\lambda^{af_i}$  as a category of graded modules over certain explicitly described graded algebra. Of course, for any

$\lambda$ , the category  $\mathcal{P}_\lambda^{af_i}$  is equivalent to the category of graded modules over the block affine Schur algebra by Proposition 4.1, but this fact is not very useful in practice, since this graded algebra is quite complicated. However, in the case of basic block the situation massively simplifies. First of all, as we observed in Lemma 5.2 we have an isomorphism of graded vector spaces

$$B_\lambda(\mathbf{k}^d \otimes A_i, \mathbf{k}^d) \simeq S_\lambda(\mathbf{k}^d \otimes A_i) \otimes S_\lambda(\mathbf{k}^{d*}).$$

However, in order to understand the multiplicative structure it is better to take a bit different point of view. Namely, by Lemma 5.2 we have decomposition

$$B_\lambda(-, \mathbf{k}^d) \simeq \bigoplus_{j=1}^{s_{\lambda,d}} S_\lambda$$

where  $s_{\lambda,d} = \dim(S_\lambda(\mathbf{k}^d))$ . Let us define a graded algebra

$$A_{i,\lambda} := \text{Ext}_{\mathcal{P}_{dp^i}}^*(S_\lambda^{(i)}, S_\lambda^{(i)}).$$

Then we have isomorphisms of graded algebras

$$B_\lambda(\mathbf{k}^d \otimes A_i, \mathbf{k}^d) \simeq \text{Ext}_{\mathcal{P}_{dp^i}}^*(B_\lambda^{(i)}(-, \mathbf{k}^d), B_\lambda^{(i)}(-, \mathbf{k}^d)) \simeq \text{Ext}_{\mathcal{P}_{dp^i}}^*\left(\bigoplus_{j=1}^{s_{\lambda,d}} S_\lambda^{(i)}, \bigoplus_{j=1}^{s_{\lambda,d}} S_\lambda^{(i)}\right) \simeq M_{s_{\lambda,d}}(A_{i,\lambda}).$$

Since any matrix algebra is Morita equivalent to the ground algebra, we obtain

**Proposition 5.5** *For any basic Young diagram  $\lambda$ , the categories  $\mathcal{P}_\lambda^{af_i}$  and  $A_{i,\lambda} - \text{grmod}$  are equivalent.*

At last, let us take a look at the graded algebra  $A_{i,\lambda}$ . Firstly, by the Collapsing Conjecture

$$A_{i,\lambda} \simeq \text{Hom}_{\mathcal{P}_d}(S_\lambda, S_{\lambda, A_i}).$$

The dimension of the latter algebra can be explicitly expressed in terms of the Littlewood–Richardson numbers. This point of view also allows one to describe the multiplication: it comes as the composite of scalar extension, Hom-multiplication and multiplication in  $A_i$ :

$$\text{Hom}_{\mathcal{P}_d}(S_\lambda, S_{\lambda, A_i}) \otimes \text{Hom}_{\mathcal{P}_d}(S_\lambda, S_{\lambda, A_i}) \longrightarrow$$

$$\mathrm{Hom}_{\mathcal{P}_d}(S_\lambda, S_{\lambda, A_i}) \otimes \mathrm{Hom}_{\mathcal{P}_d}(S_{\lambda \otimes A_i}, S_{\lambda, A_i \otimes A_i}) \longrightarrow \mathrm{Hom}_{\mathcal{P}_d}(S_\lambda, S_{\lambda, A_i \otimes A_i}) \longrightarrow \mathrm{Hom}_{\mathcal{P}_d}(S_\lambda, S_{\lambda, A_i}).$$

A bit different description of  $A_{i,\lambda}$  is perhaps even more down to earth. It follows from the fact that since  $S_\lambda$  is a direct summand in  $I^d$ , there exists an idempotent  $e_\lambda \in \mathbf{k}[\Sigma_d]$  such that  $S_\lambda = e_\lambda I^d$ . Therefore we get

$$A_{i,\lambda} \simeq e_\lambda (\mathrm{Ext}_{\mathcal{P}_{dp^i}}^*(I^{d(i)}, I^{d(i)})) e_\lambda \simeq e_\lambda (A_i^{\otimes d} \otimes \mathbf{k}[\Sigma_d]) e_\lambda.$$

A subtle point here is that even if we would take the whole  $S_\lambda$ -isotypical summand in  $I^d$  and the corresponding central idempotent  $e'_\lambda$ , this  $e'_\lambda$  is not central in the algebra  $A_i^{\otimes d} \otimes \mathbf{k}[\Sigma_d]$ . Hence  $A_{i,\lambda}$  is not Morita equivalent to a direct factor in  $A_i^{\otimes d} \otimes \mathbf{k}[\Sigma_d]$ . This is another manifestation of the fact that affine semiblocks are not genuine blocks.

All these descriptions drastically simplify for  $d = 1$ . In this case we just obtain

**Corollary 5.6** *The categories  $\mathcal{P}_1^{afi}$  and  $A_i - \mathrm{grmod}$  are equivalent.*

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